Stakeholder-Based Deterministic and Probabilistic Quality Control Using Gaussian Mixtures for Circular Factories

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Abstract. *Quality Control* (QC) in *Circular Factory*s (CFs) is fundamentally more challenging due to the uncertainty of the incoming products to be reprocessed, compared to linear factories, which have tight tolerances on the input raw materials and parts. One must have an expressive model to manage the multimodality of circular factories. QC and the decision-making process that it entails can be understood as a mapping from system requirements and variation space to discrete decisions, which can be formulated either deterministically or probabilistically. Ideally, the model must also be able to handle the deterministic and probabilistic cases seamlessly. *Gaussian Mixture Model*s (GMMs) and their universal approximation property are considered suitable for the circular factory context. GMM can also address deterministic cases by approximating the Dirac delta as a limiting case. We present a stakeholder-oriented QC framework in which stakeholder requirements, product generations, and alternative solutions for new product generations are represented probabilistically over quality-relevant variation spaces. Two complementary decision perspectives are considered. First, stakeholder requirements, exemplified by customer requirements, are modeled as acceptance regions, and the probability that a product generation satisfies individual, joint, or alternative requirements is computed by integrating the corresponding GMM density over these regions. Second, stakeholder requirements and alternative solutions are represented as probability density functions, and distance- and divergence-based metrics are used to quantify their dissimilarity. These metrics support threshold-based acceptance and ranking of alternative solutions under uncertainty. Special attention is given to deterministic limits, threshold design, and the interpretation of distances

and divergences when comparing GMMs. The proposed framework provides a basis for stakeholder-based deterministic and probabilistic QC and decision support in circular factories.

Keywords: Quality Control · Probabilistic Representation · Gaussian Mixture · Circular Factory.

1 Introduction

The CF, which reprocesses and technologically updates returned products, presents different challenges for QC compared to traditional linear production systems. In linear factories, raw materials and parts are sourced according to narrow specifications and processes are tightly controlled, allowing quality to be assessed by deterministic rules or unimodal probabilistic models. By contrast, CFs handle products with diverse and partially unknown usage histories, resulting in increased uncertainties with heterogeneous and multi-modal quality characteristics. Compared with conventional QC strategies, there is a need for more expressive and probabilistic representations to develop new product generations under uncertainty.

In this paper, we propose a framework for stakeholder-based QC in the CF, representing both stakeholder requirements and alternative solutions for new product generations as *Probability Density Function*s (PDFs) with GMMs. This enables the modeling of multi-modal populations and allows both probabilistic and deterministic information to be handled within a single mathematical structure. From a product development perspective, the consideration of alternative solutions is relevant because future product generations in the CF need to be planned with respect to returned products, reusable elements, and potential reprocessing efforts. This requires the alignment of development paths for new product generations with value-retention paths of returned products and components. Reference architecture models for the Perpetual Innovative Product provide a basis for structuring such solution spaces across product generations [9]. Here, stakeholder requirements refer to system-relevant requirements that may originate from customers, production, remanufacturing, quality assurance, or regulatory constraints [17]. The customer perspective is used in the subsequent examples as one specific stakeholder case.

A central feature of our approach is the dual use of probabilistic reasoning. First, a set of stakeholder-specific requirements, exemplified by customer requirements, can be specified as geometric regions (such as hyperrectangles or hyperellipses) in the product variation space, representing acceptable quality ranges. The probability that a variant of a future product generations, modeled as GMMs, satisfies specific customer requirements is then computed by integrating its PDF over the corresponding region. This supports graded acceptance decisions and enables the calculation of suitability probabilities for one or more customers–including intersections and unions of acceptance regions.

Second, we introduce a distributional comparison strategy, where both the customer requirements and alternative solutions for a new product generation

are modeled as GMMs, and distances or divergences (with closed-form solutions for GMMs) are used to quantify the suitability of alternative solutions. This supports both threshold-based acceptance and ranking for development decisions.

The contribution of this paper is not the introduction of GMMs as a general uncertainty representation, which was developed in [1]. Rather, the contribution is the application and extension of that representation to stakeholder-oriented QC in CFs. Specifically, we contribute:

1. a stakeholder-based decision framework in which product-generation uncertainty, stakeholder tolerances, and stakeholder target profiles are evaluated in a common variation space;
2. two explicit and complementary decision routes: region-based acceptance probabilities for tolerance-like requirements, and distribution-based divergence measures for probabilistic target profiles;
3. decision rules for individual, joint, alternative, and exclusive stakeholder acceptance;
4. a numerical example showing that region-probability and divergence-based decisions can lead to different rankings, and therefore answer different decision questions.

By combining probability calculations on stakeholder-specific acceptance regions with distribution-based comparisons, our framework enables stakeholder-oriented quality assessment and decision support for circular products. The subsequent sections detail these approaches, present relevant mathematical tools, and illustrate their application through concrete scenarios.

2 Probabilistic Representation with GMMs

This work adopts the probabilistic uncertainty representation developed in [1], but the basic notation is summarized here for self-containment. Let $x \in \mathbb{R}^d$ denote a vector of quality-relevant variation variables. A GMM density is written as

$$p(x) = \mathcal{G}(x|g) = \sum_{m=1}^{M} \pi_m \mathcal{N}(x|m_m, \Sigma_m), \qquad \pi_m \geq 0, \quad \sum_{m=1}^{M} \pi_m = 1, \tag{1}$$

where $g = \{(\pi_m, m_m, \Sigma_m)\}_{m=1}^{M}$ collects the mixture weights, means, and covariance matrices. This representation can approximate multi-modal distributions and also includes deterministic information as a limiting case when covariance matrices approach zero.

For a stakeholder acceptance region $R \subseteq \mathbb{R}^d$, the probability that a product generation represented by p_{prod} satisfies the corresponding requirement is

$$\mathbb{P}(R) = \int_R p_{\text{prod}}(x)\, \mathrm{d}x. \tag{2}$$

For general regions this integral is evaluated numerically, for example by Monte Carlo sampling,

$$\mathbb{P}(R) \approx \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_R(x^{(k)}), \qquad x^{(k)} \sim p_{\text{prod}}. \tag{3}$$

For axis-aligned rectangular regions and diagonal Gaussian covariances, the same quantity can also be computed directly using Gaussian cumulative distribution functions.

The second decision route compares two densities directly. For GMMs, many quantities of the form

$$J(p, q) = \int p(x)q(x)\,\mathrm{d}x \tag{4}$$

have closed forms. If $p(x) = \sum_i \pi_i^p \mathcal{N}(x|m_i^p, \Sigma_i^p)$ and $q(x) = \sum_i \pi_i^q \mathcal{N}(x|m_i^q, \Sigma_i^q)$, then

$$J(p, q) = \sum_i \sum_j \pi_i^p \pi_j^q \mathcal{N}(m_i^p | m_j^q, \Sigma_i^p + \Sigma_j^q). \tag{5}$$

This identity yields, for example, the squared L^2 distance

$$\|p - q\|_2^2 = J(p, p) + J(q, q) - 2J(p, q), \tag{6}$$

and the Cauchy–Schwarz divergence

$$D_{\text{CS}}(p, q) = -\log \frac{J(p, q)}{\sqrt{J(p, p)J(q, q)}}. \tag{7}$$

Other closed-form divergences for GMMs, such as the Total Squared Loss and Jensen–Quadratic Rényi divergence, can be used analogously [7]. Comprehensive treatments of mixture models and their statistical properties are given in [26,10,27,13].

3 From Ensemble PDF to Customer-Based PDF

The posterior product distribution used for decision making may be obtained by combining ensemble-level knowledge with instance-specific measurements. In the GMM setting this can be done analytically because the product of two Gaussian densities is proportional to another Gaussian. For two Gaussian factors,

$$\mathcal{N}(x|m_a, \Sigma_a)\mathcal{N}(x|m_b, \Sigma_b) = c_{ab}\mathcal{N}(x|m_{ab}, \Sigma_{ab}), \tag{8}$$

where

$$\Sigma_{ab} = \left(\Sigma_a^{-1} + \Sigma_b^{-1}\right)^{-1}, \qquad m_{ab} = \Sigma_{ab}\left(\Sigma_a^{-1} m_a + \Sigma_b^{-1} m_b\right), \tag{9}$$

and c_{ab} is the normalization factor $\mathcal{N}(m_a|m_b, \Sigma_a + \Sigma_b)$. Hence, if both the prior ensemble knowledge and the measurement likelihood are represented as GMMs,

the posterior is again a GMM obtained by forming all pairwise Gaussian products and renormalizing the resulting mixture weights. This is the Bayesian-fusion mechanism described in detail in [1].

The resulting posterior product density $p_{\text{prod}}(x) = \mathcal{G}(x|g_{\text{prod}})$ forms the basis for the two stakeholder-oriented decision strategies considered below: integration over stakeholder-specific acceptance regions and direct distributional comparison with stakeholder target densities.

3.1 Choice of Decision Strategy

The two decision routes serve different purposes. Region-based probabilities are appropriate when stakeholder requirements are given as tolerances, intervals, or admissible regions in variation space. They have a direct interpretation: $\mathbb{P}(R)$ is the probability that a product generation satisfies the requirement R.

Divergence-based comparison is useful when the requirement is not naturally a hard region but rather a target distribution, for example a desired population of product states, a preferred distribution of returned components, or a reference distribution representing a stakeholder profile. In this case the question is not only whether samples fall inside a tolerance region, but how similar the complete product distribution is to the target distribution. This can be relevant for ranking alternative solutions, comparing development paths, or estimating compatibility with the expected state distribution of incoming products.

The two criteria are therefore not equivalent. A product distribution may have high probability inside a tolerance region while still being distributionally different from the stakeholder target density. Conversely, a distribution may be close to a target density but still have non-negligible probability outside a strict acceptance region. For this reason, the choice of strategy should be based on the nature of the stakeholder requirement: tolerance-like requirements should primarily use region probabilities, whereas population-shape or target-profile requirements can be assessed using divergences.

4 Stakeholder Requirements and Decision Making

Each stakeholder is associated with a set of system requirements defining an acceptance region in variation space. In the following, customers are used as one specific stakeholder example, while the same logic can also represent requirements from production, remanufacturing, quality assurance, or regulation. Customer-oriented decisions are therefore based on evaluating the probability that a product generation satisfies one or more such regions..

In multi-stakeholder settings, decisions involve individual, joint, or alternative acceptance across stakeholder-specific requirement regions, corresponding to standard set-theoretic operations [23,5]. Figure 1 illustrates this setting for three customers as stakeholder examples, where the acceptance regions are represented as nested cubes.

Example 1 (Scenario I).

Let $A \subseteq B \subseteq C$. Then $A \cap B = A$, $A \cap B \cap C = A$, and $A \setminus B = \varnothing$ as in Figure 1. Consequently, the probabilities of a product generation, variant or alternative solution being acceptable for A, for both A and B, and for A, B, and C are respectively $\mathbb{P}(A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A)$, and $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)$. Furthermore, the probability of being acceptable for A but not for B is $\mathbb{P}(A \setminus B) = 0$.

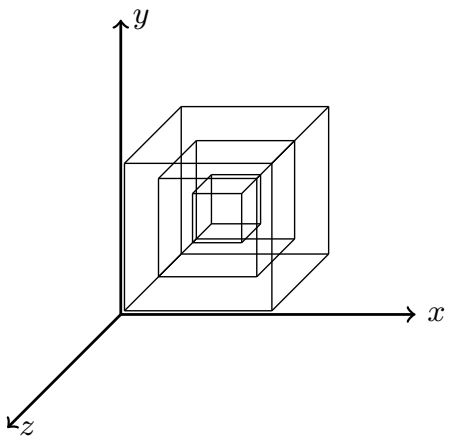


Fig. 1. Customer acceptance regions represented as nested cubes in variation space.

Example 2 (Scenario II).

Let $A \cap B \neq \varnothing$, $A \cap C \neq \varnothing$, and $B \cap C \neq \varnothing$, with possibly $A \cap B \cap C \neq \varnothing$. Then $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A \cap B \cap C)$ are determined by the probability mass contained in the corresponding overlap regions. Furthermore, $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$.

Remark 1. All probabilities in the preceding scenarios are computed over unions and intersections of hyperrectangles, i.e. logical combinations of axis-aligned constraints on x. These integrals are evaluated numerically by software using Monte Carlo sampling, where $\mathbb{P}(E) \approx \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_E(x^{(k)})$.

The underlying density is given by the GMM $\mathcal{G}(x|g_{\text{prod}})$, and all integrals are taken with respect to this measure. For unions, σ-additivity implies $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, which explains why integrals over $c_A \cup c_B$ reduce to combinations of integrals over unions of hyperrectangles where c_A and c_B are the corresponding hyperrectangles of A and B.

Consequently, expressions such as $\mathbb{P}(A) = \int_{c_A} \mathcal{G}(x|g_{\text{prod}})\mathrm{d}x$, $\mathbb{P}(A \cap B) = \int_{c_A \cap c_B} \mathcal{G}(x|g_{\text{prod}})\mathrm{d}x$, and $\mathbb{P}(A \cup B) = \int_{c_A \cup c_B} \mathcal{G}(x|g_{\text{prod}})\mathrm{d}x$ are all evaluated numerically by the same software routine.

Remark 2 (Deterministic Limit and Numerical Evaluation).

In the deterministic case, the GMM $\mathcal{G}(x|g_{\text{prod}})$ may collapse to (or approximate) a sum of Dirac measures. In this regime, integrals over regions are still evaluated by the same software, which interprets them consistently in the limiting sense. However, if acceptance regions are defined too strictly (e.g. as exact sets of measure zero under a continuous approximation), this may lead to degenerate evaluations such as $\mathbb{P}(A) = 0$ for all non-trivial events. In practice, this motivates using relaxed regions or regularized PDF to ensure meaningful numerical probabilities.

4.1 Numerical Illustration in a Two-Dimensional Variation Space

To illustrate the framework beyond symbolic set identities, we consider a two-dimensional variation space $x = (x_1, x_2)^\top$. The posterior product generation is modeled as the two-component GMM

$$\begin{aligned} p_{\text{prod}}(x) = 0.6\,\mathcal{N}\left(x \middle| \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0.25 & 0\\ 0 & 0.25\end{bmatrix}\right) \\ + 0.4\,\mathcal{N}\left(x \middle| \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}0.36 & 0\\ 0 & 0.16\end{bmatrix}\right). \end{aligned} \tag{10}$$

Three customer acceptance regions are defined as rectangles,

$$\begin{aligned} c_A &= [-0.5, 0.5] \times [-0.5, 0.5], \\ c_B &= [-1, 1] \times [-0.75, 0.75], \\ c_C &= [1.4, 2.6] \times [0.6, 1.4]. \end{aligned} \tag{11}$$

The resulting probabilities are shown in Table 1. They were computed using Gaussian cumulative distribution functions; Monte Carlo integration gives the same values up to sampling error.

Table 1. Acceptance probabilities for three rectangular customer regions.

Region	Probability under p_{prod}
c_A	0.280
c_B	0.501
c_C	0.187
$c_A \cap c_B$	0.280
$c_A \cup c_C$	0.467

The same product density can also be compared with stakeholder target densities. For illustration, let

$$\begin{aligned} p_A &= \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0.25 & 0\\ 0 & 0.25\end{bmatrix}\right), \\ p_B &= \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0.64 & 0\\ 0 & 0.64\end{bmatrix}\right), \\ p_C &= \mathcal{N}\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}0.36 & 0\\ 0 & 0.16\end{bmatrix}\right). \end{aligned} \tag{12}$$

Table 2 reports the squared L^2 distance and the Cauchy–Schwarz divergence. These values are computed in closed form using the identity

$$J(p,q) = \int p(x)q(x)\,\mathrm{d}x = \sum_i \sum_j \pi_i^p \pi_j^q \mathcal{N}(m_i^p | m_j^q, \Sigma_i^p + \Sigma_j^q). \tag{13}$$

Table 2. Closed-form distributional mismatch between the product density and three stakeholder target densities.

Target density	$\|p_{\mathrm{prod}} - p_i\|_2^2$	$D_{\mathrm{CS}}(p_{\mathrm{prod}}, p_i)$
p_A	0.103	0.188
p_B	0.069	0.255
p_C	0.231	0.565

The example shows that probability-based and divergence-based assessments answer different questions. The region probabilities in Table 1 quantify how much mass lies inside each customer tolerance region. The divergences in Table 2 instead quantify the similarity between the entire product distribution and a stakeholder target distribution. Consequently, the two assessments need not rank stakeholders or alternatives in the same order.

Example 3 (Region Probability and Divergence Can Disagree). Consider the customer region $c_A^\star = [-1,1] \times [-1,1]$ and the target density

$$p_A^\star = \mathcal{N}\left(x | \begin{bmatrix}0\\0\end{bmatrix}, 0.25I\right).$$

Compare two candidate product distributions:

$$q_1 = \mathcal{N}\left(x | \begin{bmatrix}0\\0\end{bmatrix}, 0.01I\right), \qquad q_2 = p_A^\star.$$

The first candidate is very narrow and lies almost completely inside the acceptance region. The second candidate exactly matches the stakeholder target density. Their scores are shown in Table 3.

Table 3. Example where region probability and divergence lead to different preferences.

Candidate	$\mathbb{P}(c_A^\star)$	$\|q - p_A^\star\|_2^2$	$D_{\mathrm{CS}}(q, p_A^\star)$
$q_1 = \mathcal{N}(0, 0.01I)$	≈ 1.000	7.052	0.956
$q_2 = p_A^\star$	0.911	0.000	0.000

Thus, if the stakeholder requirement is purely tolerance-based, q_1 is preferable because it almost certainly satisfies the region. If the stakeholder requirement is a target population distribution, q_2 is preferable because it exactly matches the target density. This illustrates why the two decision strategies should be regarded as complementary rather than interchangeable.

4.2 Stakeholder Suitability via Distances Between GMMs

In the distributional formulation, suitability is no longer defined via geometric acceptance regions but via comparison of PDFs. Each stakeholder-specific set of requirements i can be represented by a target quality distribution $p_i(x) = \mathcal{G}(x|g_i)$, while the product generation, variant or alternative solution is described by $p_{\mathrm{prod}}(x) = \mathcal{G}(x|g_{\mathrm{prod}})$.

Suitability is then assessed through a non-negative divergence $D(p_{\mathrm{prod}}, p_i)$, without invoking event probabilities. For each stakeholder-specific set of requirements, the scalar score $D_i := D(p_{\mathrm{prod}}, p_i)$ is computed and compared against a threshold τ_i, yielding the acceptance rule $D_i \leq \tau_i$.

In addition to stakeholder-specific acceptance, the same formulation can be used to compare alternative solutions for a new product generation. In this case, each alternative solution is represented by a PDF over quality-relevant attributes, and divergences quantify its mismatch with stakeholder requirements, target profiles, or expected states of incoming products. This enables threshold-based acceptance as well as ranking of alternative solutions under uncertainty. An alternative solution may be considered advantageous if it shows a lower mismatch with the probabilistic state distribution of returned products, since this can indicate a higher likelihood of satisfying requirements with reduced reprocessing effort.

Remark 3 (Distances and Divergences with Closed Forms for GMMs).
Numerous distances and divergences have been proposed and exist [22,24,15] for comparing PDFs. In the context of GMMs, it is advised to consider the L^2 distance [19,8,16], the Total Squared Loss (tSL)[18,11,2,3,20,21,7], the Cauchy-Schwarz divergence [25,14,12,7], and the Jensen–Quadratic Rényi (JQR) divergence [28,4,6,7] . These measures are selected because they admit closed-form expressions for GMM [7], enabling efficient evaluation without numerical integration.

Stakeholder Models and Divergence-Based Acceptance Let $p_{\text{prod}}(x) = \mathcal{G}(x|g_{\text{prod}})$ and $p_i(x) = \mathcal{G}(x|g_i)$ denote the product generation and customer GMMs. A divergence $D(p_{\text{prod}}, p_i)$, computable in closed form for GMM, yields a scalar mismatch D_i. Acceptance is defined by the boolean variable $A \coloneqq D_i \leq \tau_i$.

Remark 4 (Multi-threshold Acceptance). Multiple thresholds $0 \leq \tau_i^{(1)} < \cdots < \tau_i^{(K_i)}$ induce graded suitability levels (e.g. excellent/good/acceptable/rejected) via partitioning of the range of D_i.

Divergence-Based Scenarios for Three Customers We check whether the boolean variables $A \coloneqq D_A \leq \tau_A$,$B \coloneqq D_B \leq \tau_B$, and $C \coloneqq D_C \leq \tau_C$, where $D_i = D(p_{\text{prod}}, p_i)$ are satisfied or not.

Remark 5. No integration over regions in variation space is required; all decisions reduce to evaluating divergence inequalities and their logical combinations.

Example 4 (Product at Hand and Same Distance or Divergence Measure).
Assume $p_A \neq p_B \neq p_C \neq p_{\text{prod}}$ with $\tau_A < \tau_B < \tau_C$ which conceptually is similar to Figure 2. Hence $A \wedge B \equiv A$ and $A \wedge B \wedge C \equiv A$, and $\neg A \equiv D_A > \tau_A$, $\neg A \wedge B \equiv (\tau_A < D_A) \wedge (D_B \leq \tau_B)$, while rejection of all customers corresponds to $(D_A > \tau_C) \wedge (D_B > \tau_C) \wedge (D_C > \tau_C)$.

Remark 6. In Figure 2, the smallest square represents the set of all PDFs whose distance or divergence from the product density does not exceed τ_A. Analogously, the medium-sized square represents the set of all PDFs whose distance or divergence from the product PDF does not exceed τ_B, and the largest square corresponds to an even larger threshold.

Remark 7. In Figure 2, the metric inducing the circular level sets appears to be rotationally invariant, whereas the metric inducing the square-shaped level sets does not. This illustrates that intuitive comparisons between distances and divergences can be deceptive.

Remark 8 (On Threshold Design and Divergence Interpretation).
Threshold selection requires understanding the scale and meaning of the chosen divergence. This is analogous to asking whether two points (e.g. 2 and 3) are "close", which depends entirely on the metric: under the usual absolute value, $|2 - 3| = 1$, whereas under a discrete metric the distance is 1 regardless of magnitude. Similarly, interpreting whether $D(p_{\text{prod}}, p_i) \leq \tau_i$ indicates sufficient similarity depends on the divergence geometry and its scale relative to the context of the stakeholder requirement, target profile, or development decision.

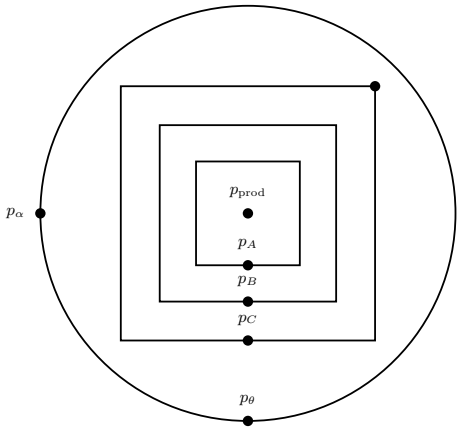


Fig. 2. The product is characterized by a given density, while different stakeholders are associated with different densities. A fixed distance or divergence measure is used to quantify the discrepancy between densities. The squares represent level sets consisting of all densities that are at the same distance or divergence from the product density with respect to a particular measure. The circle represents a level set corresponding to a different distance measure, containing all densities that are equally distanced from the product density under that measure.

Remark 9 (Practical Threshold Selection). In practice, thresholds should be chosen separately for each stakeholder and for each divergence, because different divergences have different scales. A simple calibration strategy is to compute divergence values for a reference set of historically accepted product generations or simulated acceptable alternatives and to choose τ_i as the maximum value, or as a high quantile such as the 95% quantile, of this accepted set. If rejected examples are available, τ_i may instead be selected to separate accepted and rejected cases, for example by minimizing classification error. Since divergences are sensitive to scaling, all quality variables should be normalized or expressed in physically meaningful units before thresholds are calibrated.

Remark 10 (Deterministic Limits and Strict Divergences).

In deterministic limits, GMM may collapse to Dirac measures, in which case divergences between p_{prod} and p_i are evaluated in a limiting sense. If the chosen divergence or thresholds are too strict, this may lead to degenerate values (e.g. diverging or effectively infinite distances), rendering acceptance decisions meaningless. In such cases, it is often preferable to compute distances on realizations of the random variable, or, when distributions are unimodal, to use distances between expectations as a more stable surrogate.

5 Conclusion and Outlook

This paper presented a stakeholder-oriented QC framework for the CF by modeling stakeholder requirements, product generation and its alternative solutions

using GMMs. This probabilistic approach addresses multi-modal uncertainty associated with returned products, their quality-relevant attributes, and their potential use in future product generations. By testing specific multi-customer scenarios, the framework illustrates how deterministic acceptance criteria can be extended towards probabilistic and graded decision-making under uncertainty. The framework operates with two distinct decision-making strategies. Through region-based integration, stakeholder-specific requirements, exemplified by customer requirements, are modeled as geometric zones. Evaluating overlapping or disjoint profiles across different scenarios proves that complex set-theoretic operations such as joint, alternative, or exclusive customer acceptance can be computed numerically via Monte Carlo sampling over the product GMM density. Alternatively, the distributional comparison strategy does not require spatial integration over acceptance regions. By utilizing closed-form distances and divergences in the case of GMM from Remark 3, product variants and alternative solutions for new product generations can be compared with a set of stakeholder-specific requirements using explicit inequality thresholds ($D_i \leq \tau_i$). This also enables development-oriented decisions: If an alternative solution shows a lower mismatch with the expected state distribution of incoming products, it may indicate a higher compatibility with returned products and a reduced need for reprocessing or technological updating.

The numerical illustration demonstrates that the two decision strategies are not redundant: region probabilities directly measure tolerance satisfaction, whereas divergences measure similarity to an entire stakeholder target distribution and may therefore rank candidate solutions differently.

Future work will address extensions of the framework to sequential and dynamic settings, in which inspection, remanufacturing, and development decisions are coupled over time, and to higher-dimensional variation spaces that arise in real sensor-rich environments. Despite these open questions, the results presented here indicate that GMM-based probabilistic modeling, combined with distance- and divergence-based assessment, provides a promising foundation for QC and uncertainty-aware development decision support in circular production systems.

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